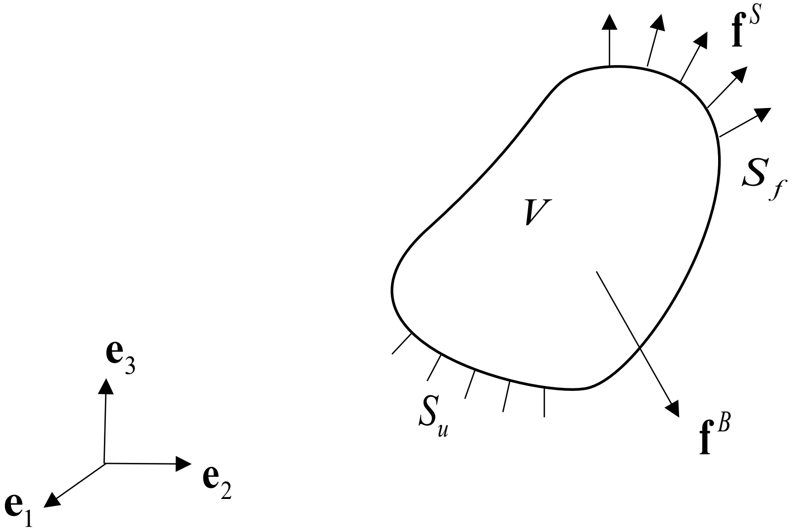
4. Finite Element Formulation



PVW: $\int_V \tau_{ij}\delta\varepsilon_{ij}dV = \int_V f_i^B \delta u_i dV + \int_{S_f} f_i^S \delta u_i dS$

Let us assume

① Small displacement: $V_o \approx V$

② Linear elastic material: $\tau_{ij} = C_{ijkl}\varepsilon_{kl}$

In the finite element formulation,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{bmatrix}, \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix},$$

where $\gamma_{ij} = 2\varepsilon_{ij} \ (i \neq j)$: engineering shear strain

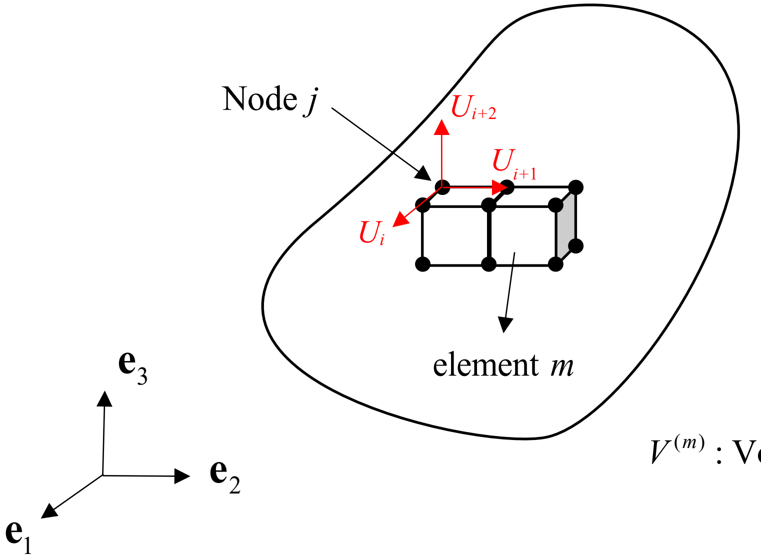
$$\tau_{ij}\varepsilon_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}\varepsilon_{ij} = \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\tau} = \boldsymbol{\varepsilon}\cdot\boldsymbol{\tau} \qquad \text{(note: } \tau_{12}\varepsilon_{12} + \tau_{21}\varepsilon_{21} = 2\tau_{12}\varepsilon_{12} = \tau_{12}\gamma_{12}\text{)}$$

Material Law: $\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\varepsilon}$

PVW in vector/matrix form:

$$\boxed{\int_V \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}\,dV = \int_V \delta\mathbf{u}^{\mathrm{T}}\mathbf{f}^B\,dV + \int_{S_f} \delta\mathbf{u}^{\mathrm{T}}\mathbf{f}^S\,dS}$$

Finite Element Discretization



$V^{(m)}$: Volume of element m, $V \approx \sum_m V^{(m)}$

Let us assume that $S_f = S$ and $S_u = \phi$ → "PVW still works."

Nodal displacement at node j → $\begin{bmatrix} U_i \\ U_{i+1} \\ U_{i+2} \end{bmatrix}$

Nodal displacement vector (nodal DOFs vector),

$$\mathbf{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_i \\ U_{i+1} \\ \vdots \\ U_N \end{bmatrix}, \qquad N\text{: number of the total DOFs}$$

PVW: $$\sum_m \int_{V^{(m)}} \delta\boldsymbol{\varepsilon}^{(m)^{\mathrm{T}}} \mathbf{C}^{(m)} \boldsymbol{\varepsilon}^{(m)} dV^{(m)} = \sum_m \int_{V^{(m)}} \delta\mathbf{u}^{(m)^{\mathrm{T}}} \mathbf{f}^{B(m)} dV^{(m)} + \sum_m \int_{S_1^{(m)} \cdots S_q^{(m)}} \delta\mathbf{u}^{s(m)^{\mathrm{T}}} \mathbf{f}^{S(m)} dS^{(m)},$$

where $S_1^{(m)} \cdots S_q^{(m)}$ are surfaces of "element m" on boundary.

Displacement Interpolations

$\mathbf{u}^{(m)} = \mathbf{H}^{(m)}\mathbf{U}$ (interpolation of displacement)

$\delta\mathbf{u}^{(m)} = \mathbf{H}^{(m)}\delta\mathbf{U}$ (interpolation of virtual displacement)

where

$\mathbf{u}^{(m)}$: displacement field of element m

$\delta\mathbf{u}^{(m)}$: virtual displacement field of element m

$\mathbf{U}$: nodal displacement vector

$\delta\mathbf{U}$: virtual nodal displacement vector

$\mathbf{H}^{(m)}$: displacement interpolation matrix for element m

(Note) The same interpolation is used for real and virtual displacements.

→ "symmetric stiffness matrix"

From the strain-displacement relation,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \rightarrow \quad \boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)}\mathbf{U}$$

$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right) \quad \rightarrow \quad \delta\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)}\delta\mathbf{U}$$

$\boldsymbol{\varepsilon}^{(m)}$: strain field of element m

$\delta\boldsymbol{\varepsilon}^{(m)}$: virtual strain field of element m

$\mathbf{B}^{(m)}$: strain interpolation matrix for element m

Using the displacement and strain interpolations in PVW, the following equation is obtained

$$\delta\mathbf{U}^{\mathrm{T}}\left[\sum_m \int_{V^{(m)}} \mathbf{B}^{(m)^{\mathrm{T}}}\mathbf{C}^{(m)}\mathbf{B}^{(m)}dV^{(m)}\right]\mathbf{U} = \delta\mathbf{U}^{\mathrm{T}}\left[\sum_m \int_{V^{(m)}} \mathbf{H}^{(m)^{\mathrm{T}}}\mathbf{f}^{B(m)}dV^{(m)} + \sum_m \int_{S_1^{(m)}\cdots S_q^{(m)}} \mathbf{H}_S^{(m)^{\mathrm{T}}}\mathbf{f}^{S(m)}dS^{(m)}\right]$$

(eq. 3.1)

This process to apply “virtual displacement vectors” is the same to

$$\cancel{\mathbf{I}^{\mathrm{T}}}\left[\sum_m \int_{V^{(m)}} \mathbf{B}^{(m)^{\mathrm{T}}} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)}\right]\mathbf{U} = \cancel{\mathbf{I}^{\mathrm{T}}}\left[\sum_m \int_{V^{(m)}} \mathbf{H}^{(m)^{\mathrm{T}}} \mathbf{f}^{B(m)} dV^{(m)} + \sum_m \int_{S_1^{(m)} \cdots S_q^{(m)}} \mathbf{H}_s^{(m)^{\mathrm{T}}} \mathbf{f}^{S(m)} dS^{(m)}\right]$$

Finally, we obtain a set of N linear equations. → "static equilibrium equations"

$$\boxed{\mathbf{KU} = \mathbf{R}},$$

where $\mathbf{K} = \sum_m \mathbf{K}^{(m)}$ with $\mathbf{K}^{(m)} = \int_{V^{(m)}} \mathbf{B}^{(m)^{\mathrm{T}}} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)}$

$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S \qquad \begin{cases} \mathbf{R}_B = \sum_m \mathbf{R}_B^{(m)} \quad \text{with } \mathbf{R}_B^{(m)} = \int_{V^{(m)}} \mathbf{H}^{(m)^{\mathrm{T}}} \mathbf{f}^{B(m)} dV^{(m)} \\ \mathbf{R}_S = \sum_m \mathbf{R}_s^{(m)} \quad \text{with } \mathbf{R}_s^{(m)} = \int_{S_1^{(m)} \cdots S_q^{(m)}} \mathbf{H}_s^{(m)^{\mathrm{T}}} \mathbf{f}^{s(m)} dS^{(m)} \end{cases}$$

$\mathbf{K}$: Stiffness matrix

$\mathbf{U}$: Nodal displacement vector

$\mathbf{R}$: Nodal force vector

Imposition of Zero-Displacement BC

Note that $\mathbf{K}$ is singular because $S_u = \phi$ is assumed.

The displacement BC is imposed by simply getting rid of the columns corresponding to zero-displacements ($U_i = 0$) and the rows corresponding to zero-virtual displacements ($\delta U_i = 0$) in the stiffness matrix $\mathbf{K}$.

Ex) When $U_3 = U_4 = 0$,

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ & K_{22} & K_{23} & K_{24} & K_{25} \\ & & K_{33} & K_{34} & K_{35} \\ & sym. & & K_{44} & K_{45} \\ & & & & K_{55} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{bmatrix} \rightarrow \begin{bmatrix} K_{11} & K_{12} & K_{15} \\ & K_{22} & K_{25} \\ sym. & & K_{55} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_5 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_5 \end{bmatrix}$$

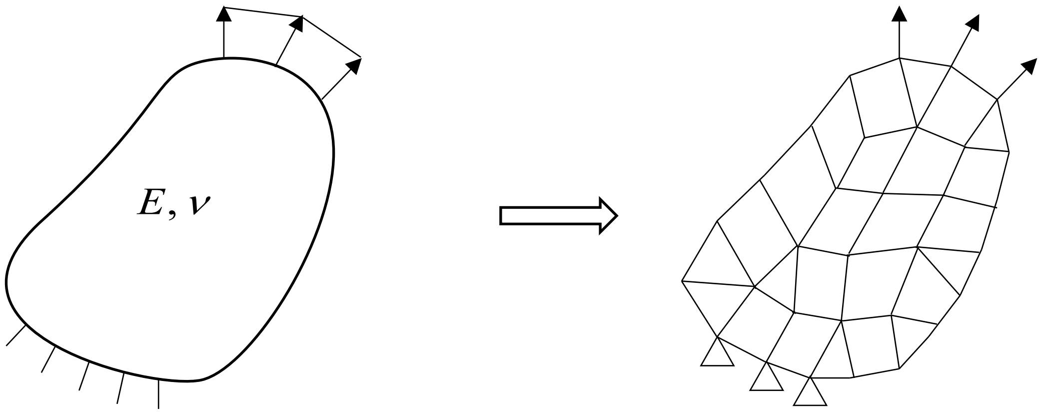
Then, $\mathbf{K}_{N\times N}$ is reduced to $\tilde{\mathbf{K}}_{\tilde{N}\times\tilde{N}}$ ($\tilde{\mathbf{N}} = \mathbf{N} -$(# of prescribed DOFs)) and the displacement and force vectors are also reduced into $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{R}}$. Finally, we get $\tilde{\mathbf{K}}\tilde{\mathbf{U}} = \tilde{\mathbf{R}}$. When the displacement BC is properly applied, the equilibrium equation can be solved.

FE Solution Procedure

Principal unknown: $\mathbf{U}$

Step 1) Geometry, material properties, applied load and displacement BC are given.

→ Construct "FE model".



FE model

FE model has information on

- nodal positions
- element connectivity (a set of nodes to construct the element)
- material properties (E, ν)
- BCs (force & displacement) are only applied at nodes

Step 2) Calculate $\mathbf{K}^{(m)}$ and $\mathbf{R}^{(m)}$ of each finite element (element matrix)

Step 3) Assemble $\mathbf{K}$ and $\mathbf{R}$ (total matrix).

$$\mathbf{K} = \sum_m \mathbf{K}^{(m)}, \quad \mathbf{R} = \sum_m \mathbf{R}^{(m)}$$

Step 4) Apply the displacement BC.

$$\mathbf{K} \rightarrow \tilde{\mathbf{K}}$$

Step 5) Solve the linear system.

$\tilde{\mathbf{K}}\tilde{\mathbf{U}} = \tilde{\mathbf{R}} \rightarrow \tilde{\mathbf{U}} = \tilde{\mathbf{K}}^{-1}\tilde{\mathbf{R}}$ → $\mathbf{U}$ is found.

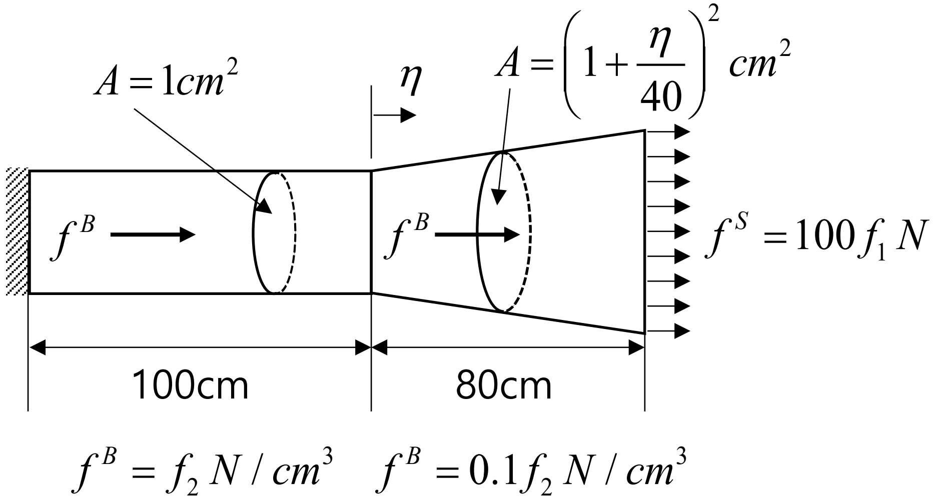
Step 6) Calculate solutions

→ Displacement field of element m: $\mathbf{u}^{(m)} = \mathbf{H}^{(m)}\mathbf{U}$

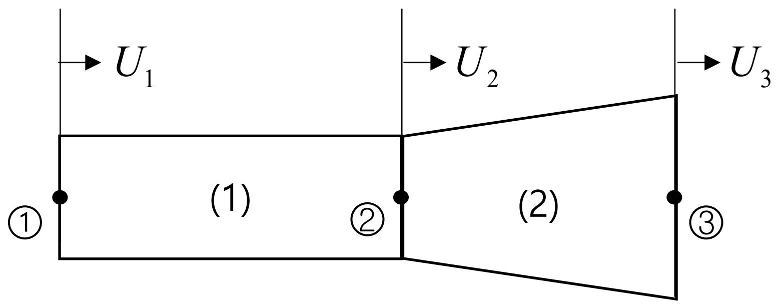
→ Strain field of element m: $\boldsymbol{\varepsilon}^{(m)} = \mathbf{B}^{(m)}\mathbf{U}$

→ Stress field of element m: $\boldsymbol{\tau}^{(m)} = \mathbf{C}^{(m)}\boldsymbol{\varepsilon}^{(m)} = \mathbf{C}^{(m)}\mathbf{B}^{(m)}\mathbf{U}$

Example – 1D bar problem

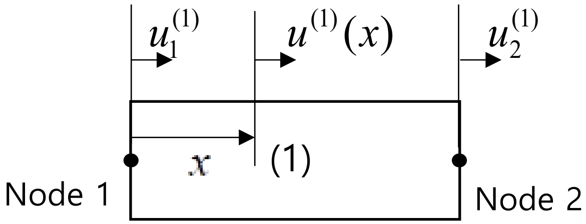


FE model

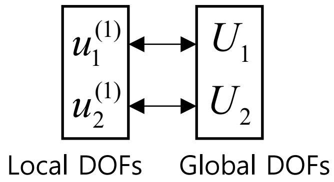


$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Element (1)



x (locally defined coordinate)

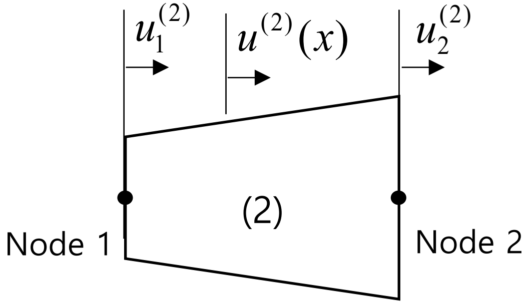


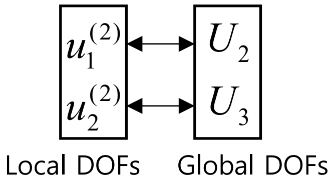
$$u^{(1)}(x) = \left(1 - \frac{x}{100}\right) u_1^{(1)} + \frac{x}{100} u_2^{(1)}$$

$$u^{(1)}(x) = \begin{bmatrix} 1 - \frac{x}{100} & \frac{x}{100} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1 - \frac{x}{100} & \frac{x}{100} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{H}^{(1)}\mathbf{U}$$

$$\varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{B}^{(1)}\mathbf{U}$$

Element (2)





$$u^{(2)}(x) = \left(1 - \frac{x}{80}\right) u_1^{(2)} + \frac{x}{80} u_2^{(2)}$$

$$u^{(2)}(x) = \begin{bmatrix} 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{H}^{(2)} \mathbf{U}$$

$$\varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = \begin{bmatrix} -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{B}^{(2)} \mathbf{U}$$

Stiffness matrix

$$\mathbf{KU}=\mathbf{R} \quad \text{with } \mathbf{U}=\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\mathbf{K}=\sum_{m=1}^{2}\mathbf{K}^{(m)}=\mathbf{K}^{(1)}+\mathbf{K}^{(2)}$$

$$=\int_{V^{(1)}}\mathbf{B}^{(1)^T}E\mathbf{B}^{(1)}dV+\int_{V^{(2)}}\mathbf{B}^{(2)^T}E\mathbf{B}^{(2)}dV^{(2)}$$

$$\mathbf{K}=\int_0^{100}1\times\begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \\ 0 \end{bmatrix}E\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix}dx+\int_0^{80}\left(1+\frac{x}{40}\right)^2\begin{bmatrix} 0 \\ -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix}E\begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix}dx$$

$$=\frac{E}{100}\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}+\frac{13E}{240}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{bmatrix}=\frac{E}{240}\begin{bmatrix}2.4 & -2.4 & 0\\ -2.4 & 15.4 & -13\\ 0 & -13 & 13\end{bmatrix}$$

Direct stiffness method

$$\mathbf{K}^{(1)} = \int_0^{100} 1 \times \begin{bmatrix} -\dfrac{1}{100} \\ \dfrac{1}{100} \end{bmatrix} E \begin{bmatrix} -\dfrac{1}{100} & \dfrac{1}{100} \end{bmatrix} dx = \frac{E}{100}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = \int_0^{80} \left(1 + \frac{x}{40}\right)^2 \times \begin{bmatrix} -\dfrac{1}{80} \\ \dfrac{1}{80} \end{bmatrix} E \begin{bmatrix} -\dfrac{1}{80} & \dfrac{1}{80} \end{bmatrix} dx = \frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \dfrac{E}{100} & -\dfrac{E}{100} & 0 \\ -\dfrac{E}{100} & \dfrac{E}{100} + \dfrac{13E}{240} & \dfrac{-13E}{240} \\ 0 & \dfrac{-13E}{240} & \dfrac{13E}{240} \end{bmatrix} = \frac{E}{240}\begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}$$

Load vector

$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S$$

$$\mathbf{R}_B = \sum_m \mathbf{R}_B^{(m)} \quad \text{with} \quad \mathbf{R}_B^{(m)} = \int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{(m)} dV^{(m)}$$

$$\mathbf{R}_S = \sum_m \mathbf{R}_S^{(m)} \quad \text{with} \quad \mathbf{R}_S^{(m)} = \int_{S_f^{(m)}} \mathbf{H}_S^{(m)T} \mathbf{f}^{S(m)} dS^{(m)}$$

$$\mathbf{R}_B = \int_0^{100} 1 \times \begin{bmatrix} 1 - \frac{x}{100} \\ \frac{x}{100} \\ 0 \end{bmatrix} f_2 \, dx + \int_0^{80} \left(1 + \frac{x}{40}\right)^2 \begin{bmatrix} 0 \\ 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} 0.1 f_2 \, dx = \frac{1}{3} \begin{bmatrix} 150 \\ 186 \\ 68 \end{bmatrix} f_2 \qquad \text{(Body force)}$$

$$\mathbf{R}_S = \mathbf{R}_S^{(1)} + \mathbf{R}_S^{(2)} = \int_{S_2} \mathbf{H}^{(2)T}\Big|_{x=80} \frac{100 f_1}{S_2} dS = \int_{S_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{100 f_1}{S_2} dS = \begin{bmatrix} 0 \\ 0 \\ 100 f_1 \end{bmatrix} \quad \text{(Surface force)}$$

$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S = \begin{bmatrix} 50 f_2 \\ 62 f_2 \\ \frac{68}{3} f_2 + 100 f_1 \end{bmatrix}$$

Equilibrium equation

$$\frac{E}{240} \begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 50 f_2 \\ 62 f_2 \\ \frac{68}{3} f_2 + 100 f_1 \end{bmatrix} \quad (\mathbf{KU} = \mathbf{R}, \ \mathbf{K} \text{ is singular.})$$

Imposition of displacement BC

$U_1 = 0$ (and $\delta U_1 = 0$)

$$\frac{E}{240}\begin{bmatrix} 15.4 & -13 \\ -13 & 13 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 62 f_2 \\ \dfrac{68}{3} f_2 + 100 f_1 \end{bmatrix} \qquad (\tilde{\mathbf{K}}\tilde{\mathbf{U}} = \tilde{\mathbf{R}})$$

Nodal displacement vector is found: $\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$

→ Displacement field: $\mathbf{u}^{(m)} = \mathbf{H}^{(m)}\mathbf{U}$

→ Strain field: $\varepsilon_{xx}^{(m)} = \mathbf{B}^{(m)}\mathbf{U}$

→ Stress field: $\tau_{xx}^{(m)} = E\mathbf{B}^{(m)}\mathbf{U}$