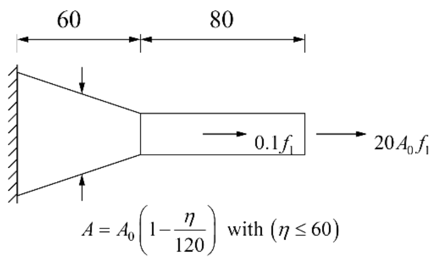
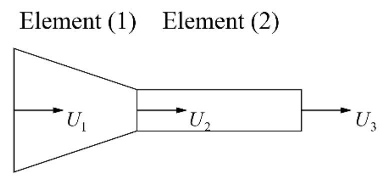
Homework #2

1.





(a)

❖ **K** matrix:

- Element (1)

$$\mathbf{u}^{(1)} = \begin{bmatrix} 1-\frac{x}{60} & \frac{x}{60} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{H}^{(1)} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \rightarrow \varepsilon_{xx}^{(1)} = \begin{bmatrix} -\frac{1}{60} & \frac{1}{60} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{B}^{(1)} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\mathbf{K}^{(1)} = \int_0^{60} \mathbf{A}^{(1)} \mathbf{B}^{(1)T} \mathbf{E} \mathbf{B}^{(1)} \mathrm{d}x = \int_0^{60} A_0 \left(1-\frac{x}{120}\right) \begin{bmatrix} -1/60 \\ 1/60 \\ 0 \end{bmatrix} E \begin{bmatrix} -\frac{1}{60} & \frac{1}{60} & 0 \end{bmatrix} \mathrm{d}x = \frac{A_0 E}{60} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Element (2)

$$\mathbf{u}^{(2)} = \begin{bmatrix} 0 & 1-\frac{x}{80} & \frac{x}{80} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{H}^{(2)} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \rightarrow \varepsilon_{xx}^{(2)} = \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \mathbf{B}^{(2)} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\mathbf{K}^{(2)} = \int_0^{80} \mathbf{A}^{(2)} \mathbf{B}^{(2)T} \mathbf{E} \mathbf{B}^{(2)} \mathrm{d}x = \int_0^{80} A_0 \begin{bmatrix} 0 \\ -1/80 \\ 1/80 \end{bmatrix} E \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix} \mathrm{d}x = \frac{A_0 E}{160} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\therefore \mathbf{K} = \mathbf{K}^{(1)} + \mathbf{K}^{(2)} = \frac{A_0 E}{160} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

❖ **R** vector:

Body force: $$\mathbf{R}_B = \int_0^{60} A_0 \left(1-\frac{x}{120}\right) \begin{bmatrix} 1-\frac{x}{60} \\ \frac{x}{60} \\ 0 \end{bmatrix} 0.1 f_1 \,\mathrm{d}x + \int_0^{80} \frac{A_0}{2} \begin{bmatrix} 0 \\ -1-\frac{x}{80} \\ \frac{x}{80} \end{bmatrix} 0.1 f_1 \,\mathrm{d}x = A_0 f_1 \begin{bmatrix} 2.5 \\ 4 \\ 2 \end{bmatrix}$$

Surface force: $$\mathbf{R}_S = 20 A_0 f_1 \begin{bmatrix} 0 \\ 1-\frac{x}{80} \\ \frac{x}{80} \end{bmatrix}_{x=80} = 20 A_0 f_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \mathbf{R} = \mathbf{R}_B + \mathbf{R}_S = A_0 f_1 \begin{bmatrix} 2.5 \\ 4 \\ 22 \end{bmatrix}$$

❖ Solve $\mathbf{U}$:

$$\mathbf{KU} = \mathbf{R} \quad \rightarrow \frac{A_0 E}{160}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = A_0 f_1 \begin{bmatrix} 2.5 \\ 4 \\ 22 \end{bmatrix}$$

$$U_1 = 0$$

$$\rightarrow \frac{A_0 E}{160}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = A_0 f_1 \begin{bmatrix} 4 \\ 22 \end{bmatrix} \quad \rightarrow \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \frac{f_1}{E}\begin{bmatrix} 2080 \\ 5600 \end{bmatrix}$$

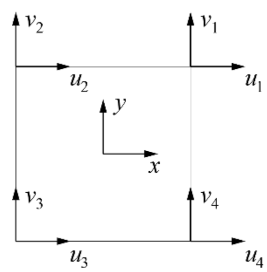
(b)

$$\mathbf{M} = \rho \int_0^{60} A_0 \left(1 - \frac{x}{120}\right) \begin{bmatrix} 1 - \frac{x}{60} \\ \frac{x}{60} \\ 0 \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{60} & \frac{x}{60} & 0 \end{bmatrix} \mathrm{d}x + \rho \int_0^{80} \frac{A_0}{2} \begin{bmatrix} 0 \\ 1 - \frac{x}{80} \\ \frac{x}{80} \end{bmatrix} \begin{bmatrix} 0 & 1 - \frac{x}{80} & \frac{x}{80} \end{bmatrix} \mathrm{d}x = \frac{\rho A_0}{6} \begin{bmatrix} 105 & 45 & 0 \\ 45 & 155 & 40 \\ 0 & 40 & 80 \end{bmatrix}$$

2.

(a)

❖ First way



$$u = \underbrace{\frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right)}_{h_1} u_1 + \underbrace{\frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right)}_{h_2} u_2 + \underbrace{\frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right)}_{h_3} u_3 + \underbrace{\frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right)}_{h_4} u_4$$

$$v = \underbrace{\frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right)}_{h_1} v_1 + \underbrace{\frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right)}_{h_2} v_2 + \underbrace{\frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right)}_{h_3} v_3 + \underbrace{\frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right)}_{h_4} v_4$$

$$\rightarrow \mathbf{H} = \begin{bmatrix} \mathbf{H}_u \\ \mathbf{H}_v \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$$

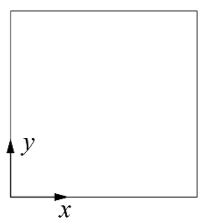
$$= \left[\begin{matrix} \frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right) & \frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right) & \frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right) & \frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right) \\ 0 & 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ \frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right) & \frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 + \frac{2}{3}y\right) & \frac{1}{4}\left(1 - \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right) & \frac{1}{4}\left(1 + \frac{x}{2}\right)\left(1 - \frac{2}{3}y\right) \end{matrix}\right]$$

$$\rightarrow \mathbf{B} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} & \frac{\partial h_3}{\partial y} & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} & \frac{\partial h_3}{\partial y} & \frac{\partial h_4}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8}\left(1+\frac{2}{3}y\right) & -\frac{1}{8}\left(1+\frac{2}{3}y\right) & -\frac{1}{8}\left(1-\frac{2}{3}y\right) & \frac{1}{8}\left(1-\frac{2}{3}y\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6}\left(1+\frac{x}{2}\right) & \frac{1}{6}\left(1-\frac{x}{2}\right) & -\frac{1}{6}\left(1-\frac{x}{2}\right) & -\frac{1}{6}\left(1+\frac{x}{2}\right) \\ \frac{1}{6}\left(1+\frac{x}{2}\right) & \frac{1}{6}\left(1-\frac{x}{2}\right) & -\frac{1}{6}\left(1-\frac{x}{2}\right) & -\frac{1}{6}\left(1+\frac{x}{2}\right) & \frac{1}{8}\left(1+\frac{2}{3}y\right) & -\frac{1}{8}\left(1+\frac{2}{3}y\right) & -\frac{1}{8}\left(1-\frac{2}{3}y\right) & \frac{1}{8}\left(1-\frac{2}{3}y\right) \end{bmatrix}$$

- Second way



$$h_1 = \frac{1}{4}\left(\frac{x}{2}\right)\left(\frac{2}{3}y\right) \quad \leftarrow (x,y)=(4,3)$$

$$h_2 = \frac{1}{4}\left(2-\frac{x}{2}\right)\left(\frac{2}{3}y\right) \quad \leftarrow (x,y)=(0,3)$$

$$h_3 = \frac{1}{4}\left(2-\frac{x}{2}\right)\left(2-\frac{2}{3}y\right) \quad \leftarrow (x,y)=(0,0)$$

$$h_4 = \frac{1}{4}\left(\frac{x}{2}\right)\left(2-\frac{2}{3}y\right) \quad \leftarrow (x,y)=(4,0)$$

(b)

- **K** matrix

$$K_{U_2U_2} = K^{(1)}_{u_4u_4} + K^{(2)}_{u_3u_3}$$

$$= t\int_{-1.5}^{1.5}\int_{-2}^{2}\left\{ E\begin{bmatrix} \frac{1}{8}\left(1-\frac{2}{3}y\right) & 0 & -\frac{1}{6}\left(1+\frac{x}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{8}\left(1-\frac{2}{3}y\right) \\ 0 \\ -\frac{1}{6a}\left(1+\frac{x}{2}\right) \end{bmatrix} \right.$$

$$\left. + E\begin{bmatrix} -\frac{1}{8}\left(1-\frac{2}{3}y\right) & 0 & -\frac{1}{6}\left(1-\frac{x}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{8a}\left(1-\frac{2}{3}y\right) \\ 0 \\ -\frac{1}{6}\left(1-\frac{x}{2}\right) \end{bmatrix} \right\} \mathrm{d}x\mathrm{d}y$$

$$= Et\int_{-1.5}^{1.5}\int_{-2}^{2}\left\{ \frac{1}{32}\left(1-\frac{2}{3}y\right)^2 + \frac{1}{36}\left(1+\frac{x^2}{4}\right) \right\} \mathrm{d}x\mathrm{d}y = \frac{17}{18}Et$$

$$K_{U_6U_7} = K^{(2)}_{u_4v_4} + K^{(3)}_{u_3v_3}$$

$$= t\int_{-1.5}^{1.5}\int_{-2}^{2}\left\{E\begin{bmatrix}0 & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{\partial h_4}{\partial x}\\ 0\\ \frac{\partial h_4}{\partial y}\end{bmatrix} + E\begin{bmatrix}0 & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{\partial h_3}{\partial x}\\ 0\\ \frac{\partial h_3}{\partial y}\end{bmatrix}\right\}\mathrm{d}x\mathrm{d}y$$

$$= Et\int_{-1.5}^{1.5}\int_{-2}^{2}\left\{-\frac{1}{96}\left(1+\frac{x}{2}\right)\left(1-\frac{2}{3}y\right)+\frac{1}{96}\left(1-\frac{x}{2}\right)\left(1-\frac{2}{3}y\right)\right\}\mathrm{d}x\mathrm{d}y = 0$$

$$K_{U_7U_6} = K_{U_6U_7} = 0$$

$$K_{U_5U_{12}} = 0$$

(c)

$$R_{B,U_9} = R^{(2)}_{B,v_1} + R^{(3)}_{B,v_2}$$

$$R_{S,U_9} = R^{(3)}_{S,v_2}$$

❖ Element (2)

$$\rightarrow \mathbf{R}^{(2)}_B = \int_{V^{(1)}} \mathbf{H}^{T(2)}\vec{f}_B\,\mathrm{d}V = -\rho g t\int_{-1.5}^{1.5}\int_{-2}^{2}\begin{bmatrix}0\\0\\0\\0\\h_1\\h_2\\h_3\\h_4\end{bmatrix}\mathrm{d}x\mathrm{d}y = -3\rho g t\begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{bmatrix} = \vec{R}^{(1)}_B = \vec{R}^{(3)}_B$$

$$\therefore R_{B,U_9} = -3\rho g t - 3\rho g t = -6\rho g t$$

$$\mathbf{R}^{(3)}_S = \int_S \mathbf{H}^{S(3)}_v \vec{f}^{\,S}\,dS = \int_S \mathbf{H}_v\Big|_{y=1.5}\left(-\frac{3}{4}x-\frac{7}{2}\right)dS = t\int_{-2}^{2}\begin{bmatrix}0\\0\\0\\0\\\frac{1}{2}\left(1+\frac{x}{2}\right)\\\frac{1}{2}\left(1-\frac{x}{2}\right)\\\frac{1}{2}\left(1-\frac{x}{2}\right)\\\frac{1}{2}\left(1+\frac{x}{2}\right)\end{bmatrix}\left(-\frac{3}{4}x-\frac{7}{2}\right)dx$$

$$\rightarrow R_{S,U_4} = R^{(3)}_{S,v_2} = t\int_{-2}^{2}\frac{1}{2}\left(1-\frac{x}{2}\right)\left(-\frac{3}{4}x-\frac{7}{2}\right)dx = -6t$$

$$\therefore R_9 = -6\rho g t - 6t$$