Homework #3

1.

(a)

Jacobian: $\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial s} \end{bmatrix}$

The coordinates of any point inside the element are interpolated from:

$$\begin{cases} x = \sum_{i=1}^{4} h_i(r,s) x_i \\ y = \sum_{i=1}^{4} h_i(r,s) y_i \end{cases} \quad \text{with,} \quad \begin{cases} h_1 = \frac{1}{4}(1+r)(1+s) \\ h_2 = \frac{1}{4}(1-r)(1+s) \\ h_3 = \frac{1}{4}(1-r)(1-s) \\ h_4 = \frac{1}{4}(1+r)(1-s) \end{cases}$$

$(x_i, y_i) = \{(3,3), (-5,5), (-3,-3), (2,-1)\}$

Derivatives of Shape Functions with respect to r and s:

$$\rightarrow \begin{cases} \dfrac{\partial h_1}{\partial r} = \frac{1}{4}(1+s) \\ \dfrac{\partial h_2}{\partial r} = -\frac{1}{4}(1+s) \\ \dfrac{\partial h_3}{\partial r} = -\frac{1}{4}(1-s) \\ \dfrac{\partial h_4}{\partial r} = \frac{1}{4}(1-s) \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial h_1}{\partial s} = \frac{1}{4}(1+r) \\ \dfrac{\partial h_2}{\partial s} = \frac{1}{4}(1-r) \\ \dfrac{\partial h_3}{\partial s} = -\frac{1}{4}(1-r) \\ \dfrac{\partial h_4}{\partial s} = -\frac{1}{4}(1+r) \end{cases}$$

Derivatives of Shape Functions at $(r = s = 0)$

$$\rightarrow \begin{cases} \dfrac{\partial h_1}{\partial r} = \frac{1}{4} \\ \dfrac{\partial h_2}{\partial r} = -\frac{1}{4} \\ \dfrac{\partial h_3}{\partial r} = -\frac{1}{4} \\ \dfrac{\partial h_4}{\partial r} = \frac{1}{4} \end{cases} \quad \begin{cases} \dfrac{\partial h_1}{\partial s} = \frac{1}{4} \\ \dfrac{\partial h_2}{\partial s} = \frac{1}{4} \\ \dfrac{\partial h_3}{\partial s} = -\frac{1}{4} \\ \dfrac{\partial h_4}{\partial s} = -\frac{1}{4} \end{cases}$$

$$\rightarrow \mathbf{J}(0,0) = \begin{bmatrix} \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial r} x_i & \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial r} y_i \\ \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial s} x_i & \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial s} y_i \end{bmatrix} = \begin{bmatrix} \dfrac{13}{4} & 0 \\ -\dfrac{1}{4} & 3 \end{bmatrix}$$

(b)

$$\mathbf{J}^{-1}\begin{bmatrix}\frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r}\\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s}\end{bmatrix}=\begin{bmatrix}a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\end{bmatrix} \quad \rightarrow \mathbf{B}=\begin{bmatrix}a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 & b_4\\ b_1 & b_2 & b_3 & b_4 & a_1 & a_2 & a_3 & a_4\end{bmatrix}$$

$$\therefore \mathbf{J}^{-1}_{(0,0)}\begin{bmatrix}\frac{\partial h_i}{\partial r}(0,0)\\ \frac{\partial h_i}{\partial s}(0,0)\end{bmatrix}=\frac{4}{39}\begin{bmatrix}3 & 0\\ \frac{1}{4} & \frac{13}{4}\end{bmatrix}\begin{bmatrix}\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\end{bmatrix}=\frac{1}{39}\begin{bmatrix}3 & -3 & -3 & 3\\ \frac{7}{2} & 3 & -\frac{7}{3} & -3\end{bmatrix}$$

$$\mathbf{B}(0,0)=\frac{1}{39}\begin{bmatrix}3 & -3 & -3 & 3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{7}{2} & 3 & -\frac{7}{2} & -3\\ \frac{7}{2} & 3 & -\frac{7}{2} & -3 & 3 & -3 & -3 & 3\end{bmatrix}$$

(c)

$$K_{u_1v_1}=t\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}(:,1)^T\,\mathbf{C}\mathbf{B}(:,5)\det(\mathbf{J})\,\mathrm{d}r\,\mathrm{d}s$$

Gaussian integration:

$$\rightarrow K_{u_1v_1}=tw_rw_s\mathbf{B}^T_{(:,1)}(0,0)\ \mathbf{C}\ \mathbf{B}_{(:,5)}(0,0)\ \det(\mathbf{J}(0,0))$$

$$=1\times 4\times\frac{1}{39}\begin{bmatrix}3 & 0 & \frac{7}{2}\end{bmatrix}\begin{bmatrix}E & 0 & 0\\ 0 & E & 0\\ 0 & 0 & E/2\end{bmatrix}\frac{1}{39}\begin{bmatrix}0\\ \frac{7}{2}\\ 3\end{bmatrix}\times\frac{39}{4}$$

$$=\frac{E}{78}\begin{bmatrix}3 & 0 & \frac{7}{4}\end{bmatrix}\begin{bmatrix}0\\ \frac{7}{2}\\ 3\end{bmatrix}=\frac{7}{52}E$$

2.

(a)

$$\begin{cases}h_1=r\\ h_2=s\\ h_3=1-r-s\end{cases}\quad\rightarrow\begin{cases}\frac{\partial h_1}{\partial r}=1 & \frac{\partial h_1}{\partial s}=0\\ \frac{\partial h_2}{\partial r}=0 & \frac{\partial h_2}{\partial s}=1\\ \frac{\partial h_3}{\partial r}=-1 & \frac{\partial h_3}{\partial s}=-1\end{cases}$$

$$(x_i,y_i)=\{(20,3),(0,10),(0,0)\}$$

$$\mathbf{J}=\begin{bmatrix}\sum_{i=1}^{3}\frac{\partial h_i}{\partial r}x_i & \sum_{i=1}^{3}\frac{\partial h_i}{\partial r}y_i\\ \sum_{i=1}^{3}\frac{\partial h_i}{\partial r}x_i & \sum_{i=1}^{3}\frac{\partial h_i}{\partial s}y_i\end{bmatrix}=\begin{bmatrix}20 & 0\\ 0 & 10\end{bmatrix}$$

(b)

$$\mathbf{J}^{-1}\begin{bmatrix} \dfrac{\partial h_1}{\partial r} & \dfrac{\partial h_2}{\partial r} & \dfrac{\partial h_3}{\partial r} \\ \dfrac{\partial h_1}{\partial s} & \dfrac{\partial h_2}{\partial s} & \dfrac{\partial h_3}{\partial s} \end{bmatrix} = \frac{1}{200}\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \frac{1}{200}\begin{bmatrix} 10 & 0 & -10 \\ 0 & 20 & -20 \end{bmatrix}$$

Local $\mathbf{B}$: $\mathbf{B}^{(1)} = \dfrac{1}{200}\begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & -20 \\ 0 & 20 & -20 & 10 & 0 & -10 \end{bmatrix}$

Global $\mathbf{B}$: $\mathbf{B} = \dfrac{1}{200}\begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix}$

(c)

Gauss integration:

$$\begin{aligned}
\mathbf{K} &= \int_V \mathbf{B}^T\mathbf{C}\mathbf{B}\det(\mathbf{J})\ \mathrm{d}V \\
&= \frac{1}{2}t\sum_{i=1}^{3}\mathbf{B}^T(r_i,s_i)\mathbf{C}\mathbf{B}(r_i,s_i)\det(\mathbf{J}(r_i,s_i))\cdot\omega_i \\
&= \frac{1}{2}t\sum_{i=1}^{3}\frac{1}{200}\begin{bmatrix} 10 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E/2 \end{bmatrix}\frac{1}{200}\begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix}\cdot 200\cdot\omega_i \\
&= \frac{E}{400}\left\{\frac{1}{3}\begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix}\right\} = E\begin{bmatrix} 1/4 & 0 \\ 0 & 1/8 \end{bmatrix}
\end{aligned}$$

Direct integration:

$$\mathbf{K} = t\int_0^1\int_0^{1-r}\mathbf{B}^T\mathbf{C}\mathbf{B}\det(\mathbf{J})\,\mathrm{d}s\mathrm{d}r = \frac{E}{200}\int_0^1\int_0^{1-r}\begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix}\mathrm{d}s\mathrm{d}r = \frac{E}{200}\begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix}\int_0^1(1-r)\mathrm{d}r = E\begin{bmatrix} 1/4 & 0 \\ 0 & 1/8 \end{bmatrix}$$